Algebraic Models of Plane Couette Equilibria

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Introduction

Invariant solutions are thought to organize the flow of turbulent trajectories of the Navier–Stokes equations. Historically, they have been computed using high-dimensional **direct numerical simulations** (DNS). We present a novel approach for computing solutions for plane Couette flow using Galerkin projection onto a divergence-free basis.

Equations and Geometry

We consider **incompressible plane Couette flow** with periodic domain in x and z and no-slip boundary conditions. In this domain, the nondimensionalized Navier-Stokes equations for the fluctuation $\vec{u}$ are written:

$$\frac{\partial \vec{u}}{\partial t} + v\,\vec{e}_x + y\frac{\partial \vec{u}}{\partial x} + \vec{u}\cdot\vec{\nabla}\vec{u} = -\vec{\nabla}p + \frac{1}{Re}\nabla^2\vec{u},$$

$$\vec{\nabla}\cdot\vec{u} = 0.$$

where Re is the **Reynolds number**, a measure of the viscosity. We further enforce **symmetries**.

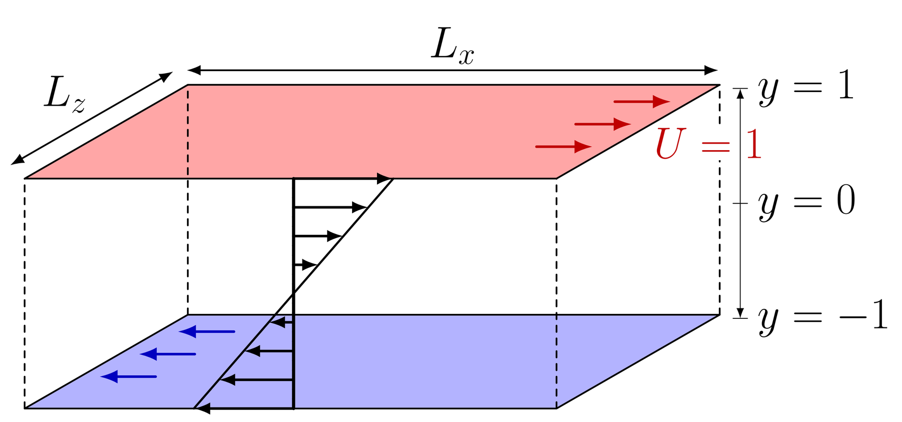

Figure 1. Diagram of plane Couette flow.

Galerkin Projection

We use a basis set $\{\vec{\Psi}_{ijkl}\}$, where each element is divergence-free and obeys boundary conditions and **symmetries**. We specify a discretization by setting parameters, (J, K, L). We then re-express the basis with a linear index and approximate velocity:

$$\vec{u}(\vec{x}, t) \approx \sum_{n=1}^{m} x_n(t)\,\vec{\Psi}_n(\vec{x}),$$

Projecting Navier–Stokes onto this basis converts the PDE into a finite-dimensional ODE,

$$B\frac{dx}{dt} = Ax + N(x).$$

Equilibria are found by solving $Ax + N(x) = 0$ in a much smaller state space than DNS.

Completeness of $\{\vec{\Psi}_{ijkl}\}$

The basis $\{\vec{\Psi}_{ijkl}\}$ is complete, meaning that for every possible $\vec{u}$ and every $\varepsilon > 0$, there exist finitely many indices and coefficients such that

$$\left\|\vec{u} - \sum_n c_n \vec{\Psi}_n\right\|_{H^1} < \varepsilon.$$

Graphically, this looks like **convergence to DNS** results as resolution improves.

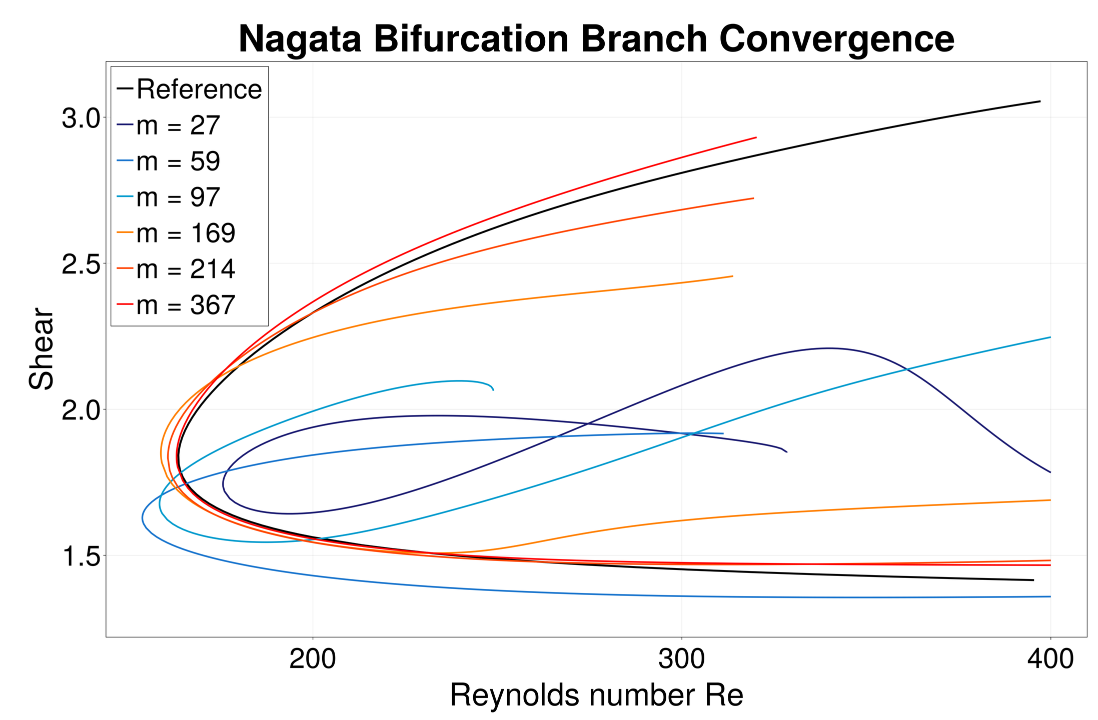

Figure 2. Bifurcation curves converge toward DNS as basis resolution increases.

Computing Equilibria

The workflow is *Galerkin projection → ODE search → DNS promotion.*

- **Low-dimensional ODE models** make large searches cheap.
- **Capture the key dynamics** of the flow at low computational cost.
- **ODE solutions lead to DNS solutions**, providing an excellent workflow for uncovering new invariants.
- **Dozens of invariant solutions** have been computed, some new to the literature.

Statistic	Value
Random seeds tested	2.80×10^6
Hookstep converged	392,943 (14.0%)
Postfilter accepted	65,995 (2.36%)
Raw unique ODE hits	588
Final unique physical equilibria	79

Table 1. Summary of statistics from random initial guesses across different symmetry groups and box sizes.

Geometry Exploration

Because the reduced model is inexpensive, we can scan box geometry (L_x, L_z) and ask where coherent structures appear at the lowest Reynolds number. This addresses an open gap in the literature.

- Heatmaps of **critical Reynolds number** reveal favorable parameter regions.
- Current scans highlight a neighborhood near $L_x \approx 10,\ L_z \approx 6.$

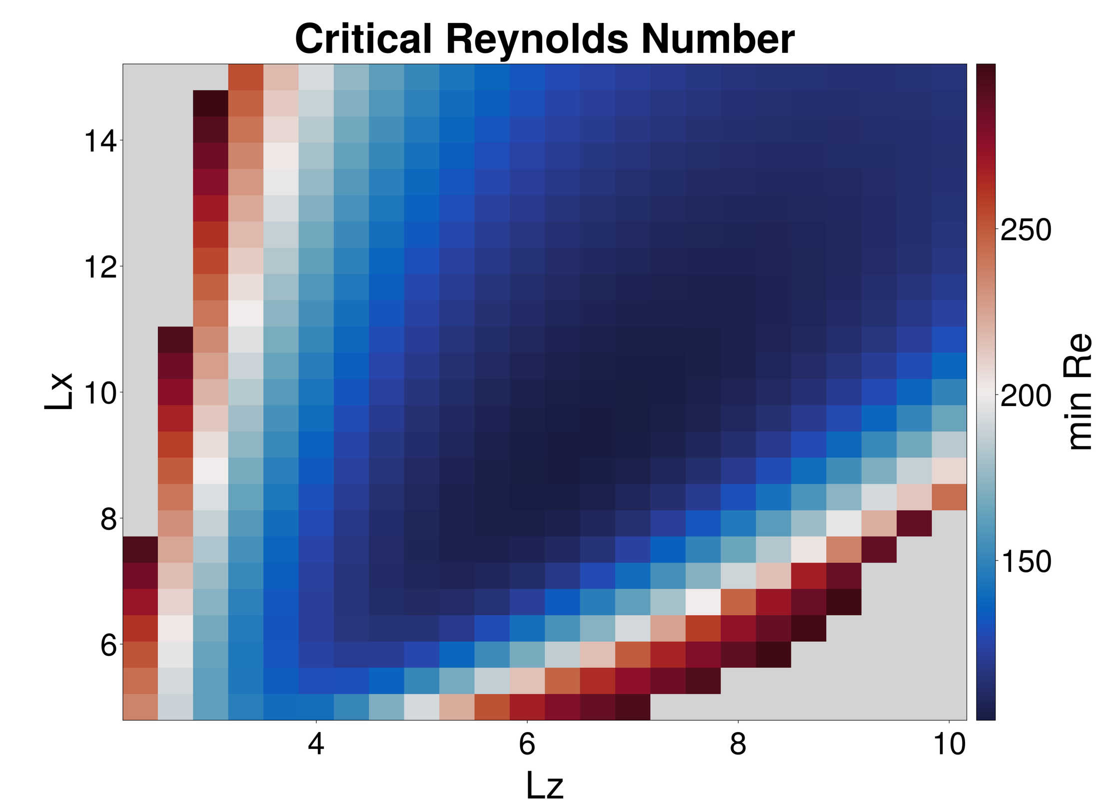

Figure 3. Heatmap of critical Reynolds number vs. geometry.

Dynamics

- Trajectory-tracking **error drops sharply** as ODE resolution increases.
- This allows for **fast exploration of dynamics**.

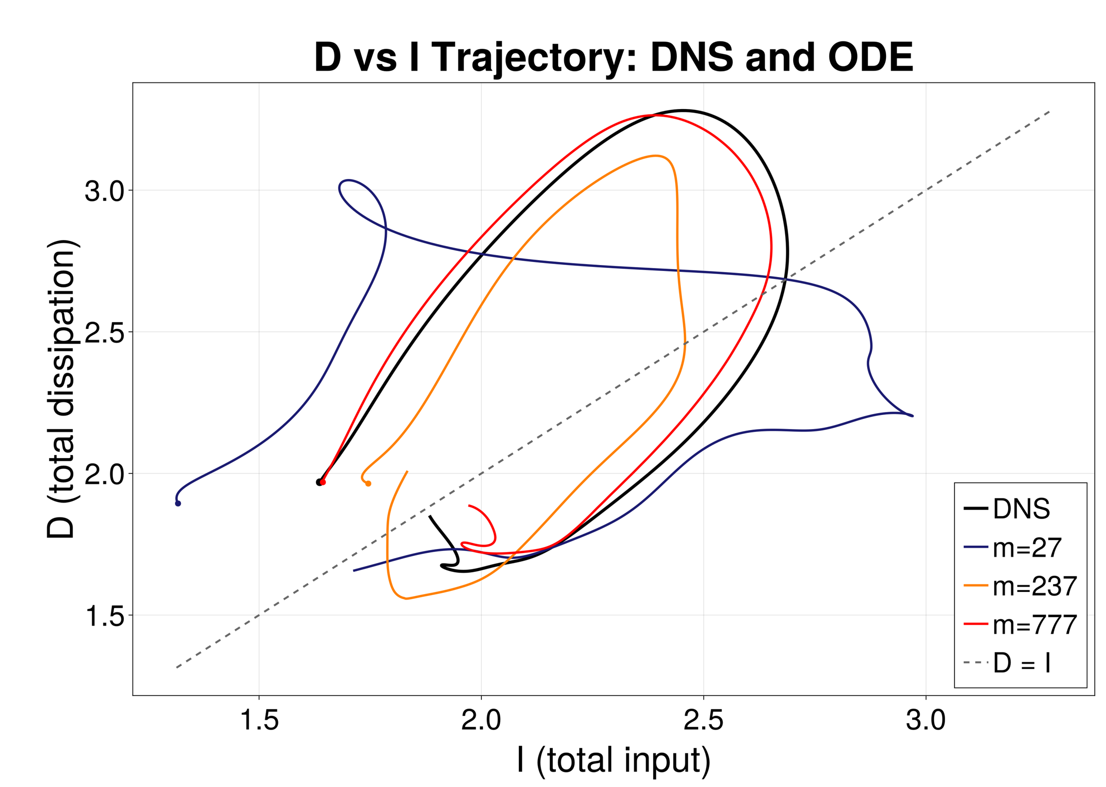

Figure 4. Dynamic convergence of ODE trajectories to DNS.

Visualizing Flowfields

DNS-vs-ODE comparisons test whether the reduced model recovers the large-scale geometry of full equilibrium solutions.

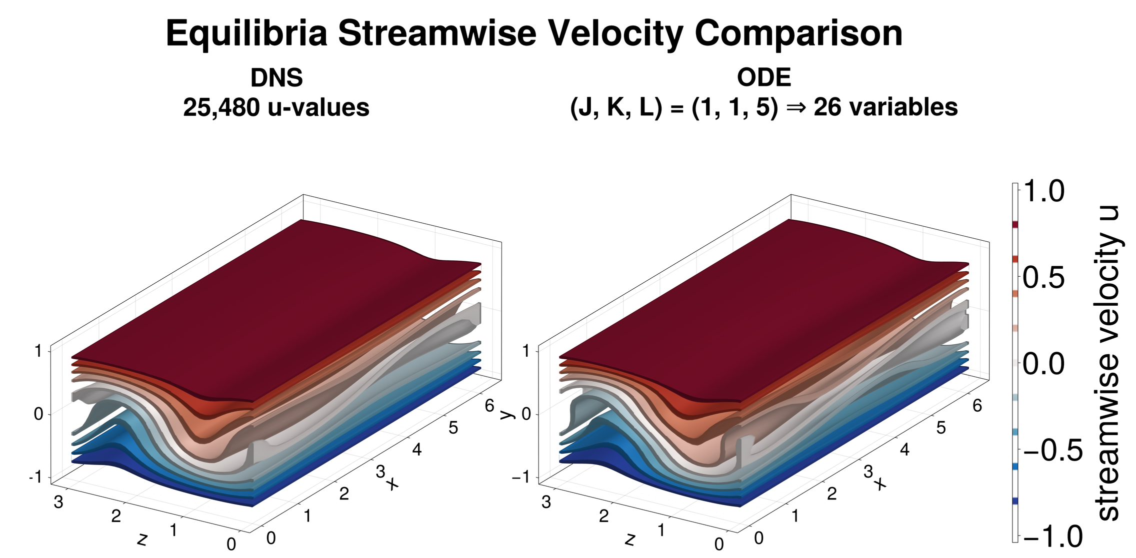

Figure 5. DNS vs. ODE comparison for a representative equilibrium at $Re = 300,\ L_x = 2\pi,\ L_z = \pi$.

Conclusion

- A divergence-free, no-slip basis gives a physically faithful Galerkin model.
- The basis is complete and converges toward DNS as resolution increases.
- We use this framework to compute equilibria, compare flowfields, and map geometry efficiently.

Key Takeaway

Our divergence-free basis approach allows fast exploration of invariant solutions to the Navier–Stokes equations for plane flow.

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