How a Candle Burns: Understanding Combustion Dynamics Using Free Boundary Problems and Phase Change

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Overview

This project investigates the usability of a simplified theoretical mathematical model intended to mimic qualitative features of combustion. To achieve this, the model will:

- Be a **self-contained** mathematical framework
- Consist of a **dynamical system** undergoing **phase change** and **material consumption**
- **Characterize steady-state solutions** where material profile and consumption rates are **constant in time**
- Arrive at a system which **mimics features of a candle**
- Introduce students **to free-boundary value problems** and **multi-phase physical models**

Methodology

Method of Characteristics: Reduction of a Partial Differential Equation (PDE) to a system of Ordinary Differential Equations (ODEs) by using its **characteristic curve**

Given a PDE of the function $U(x(t),t)$:

$$U_t + \alpha U_x = k$$

with an initial condition $U(x,0) = f(x)$ for some real number k and α is a scalar/function of x and t, a time derivative is taken to obtain:

$$\frac{dU}{dt} = U_x \frac{dx}{dt} + U_t$$

where $\boldsymbol{\frac{dx}{dt} = c}$ represents the family of **characteristic curves** of U.

Shock Discontinuities
When characteristic curves intersect one another, a **shock** is created at the point of their intersection.

These shocks represent **moving discontinuities** in the ODE's solutions.

In this case, the **PDE does not possess a smooth solution** for all times *t.*

Mathematical Modeling

A simplified model of this scenario starts with the **one-dimensional Heat Equation** and the assumption that liquid and solid phases have **equivalent thermal diffusivities** $\alpha = 1$:

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}, \qquad \alpha = \alpha_{liquid} = \alpha_{solid} = 1$$

Subject to the initial conditions:

$$x(t=0) = 0 \qquad (1)$$
$$U(x,0) = f(x) \qquad (2)$$

and boundary conditions:

$$U_x(x=t,t) = k \qquad (3)$$
$$U(s(t),t) = 0 \qquad (4)$$
$$\lim_{x\to\infty} U_x(x,t) = 0 \ \forall t \qquad (5)$$

Where $U(x(t),t)$ represents the heat of the material at a given distance x from the heat source and time t, and $s(t)$ in (4) represents the speed of the phase change boundary.

Change of Variables
We can perform a **change of variables** on the PDE to account for **material consumption**. Using the following new variables, we can examine the model in a **moving frame of reference**:

$$\xi = x - ct \quad \text{(spatial variable)}$$
$$\tau = t \quad \text{(time variable)}$$

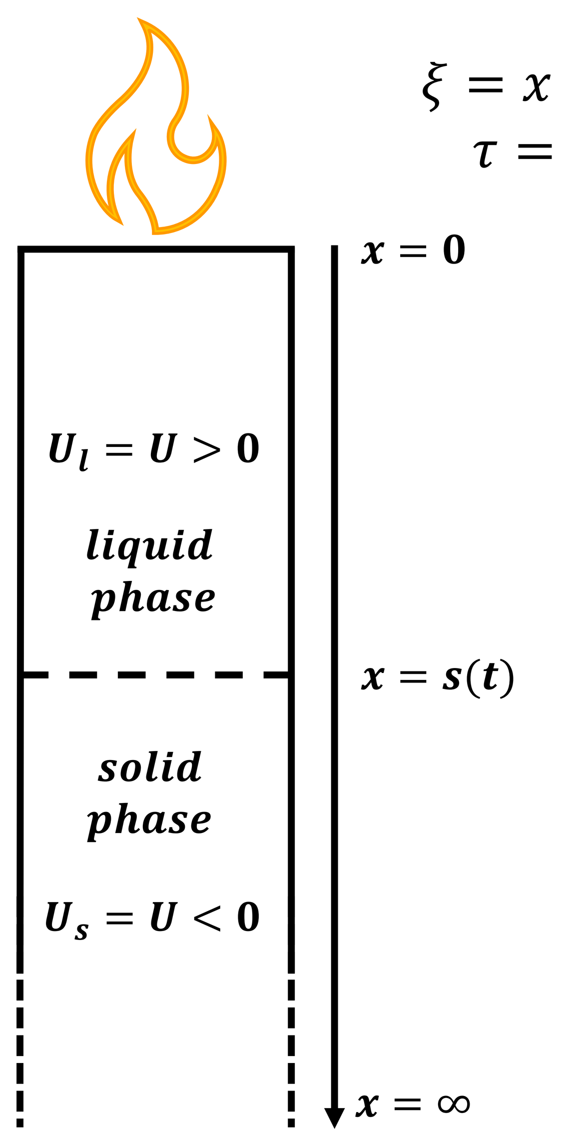

Figure 1. Physical model of the two-phase Stefan Problem.

Where c is the rate of consumption.

Using chain rule, we obtain a **steady-state** version of the **heat equation**:

$$\boxed{-cU_\xi = U_{\xi\xi}}$$

We also obtain a boundary condition **in the moving frame** now rendered as a **stationary Neumann BC**:

$$\boxed{U_x(x = ct, t) = k}$$

Thus, we now have an ODE **only dependent on** $\boldsymbol{\xi}$ that follows a moving frame of reference.

Classical Neumann Solution

Imposing Neumann boundary conditions (BCs) on the following dimensionless, steady-state, two-phase Stefan problem gives more insight into the PDE's behavior.

$$\frac{\partial u}{\partial t} = \frac{1}{\mathrm{St}} \cdot \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < s(t), \qquad t > 0$$
$$\frac{\partial u}{\partial t} = \frac{\kappa}{\mathrm{St}} \cdot \frac{\partial^2 u}{\partial x^2} \qquad s(t) < x < 1, \qquad t > 0$$

Subject to Neumann BCs for $t > 0$:

$$u(0,t) = 1, \qquad u_x(1,t) = 0$$

As $\mathrm{St} \to 0$, the following steady-state solution is obtained:

$$u(x,t) = \begin{cases} 1 - \frac{x}{s(t)} & x < s(t) \\ 0 & s(t) < x \end{cases} \qquad \text{with } s(t) = \sqrt{2t}$$

Note that κ is a dimensionless heat constant and $\mathrm{St} = c \cdot \frac{T_1 - T_m}{L}$ is the dimensionless *Stefan Number* which depends on heat capacity c, initial and melting temperatures T_1 & T_m, and latent heat L.

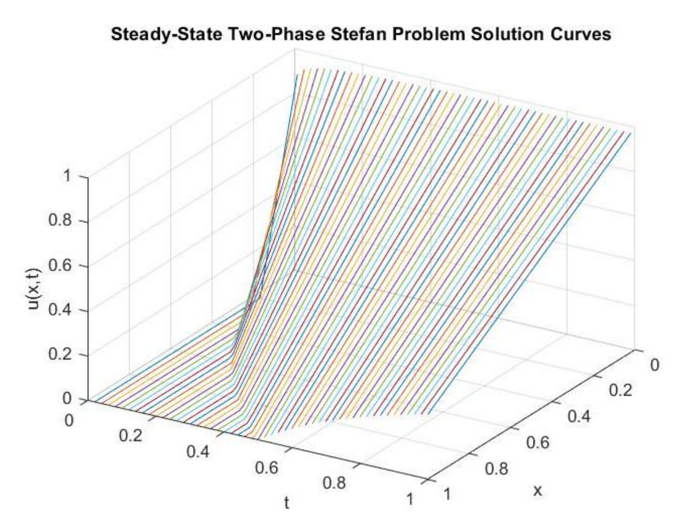

Figure 2. Solution curves to the Stefan Problem in three dimensions.

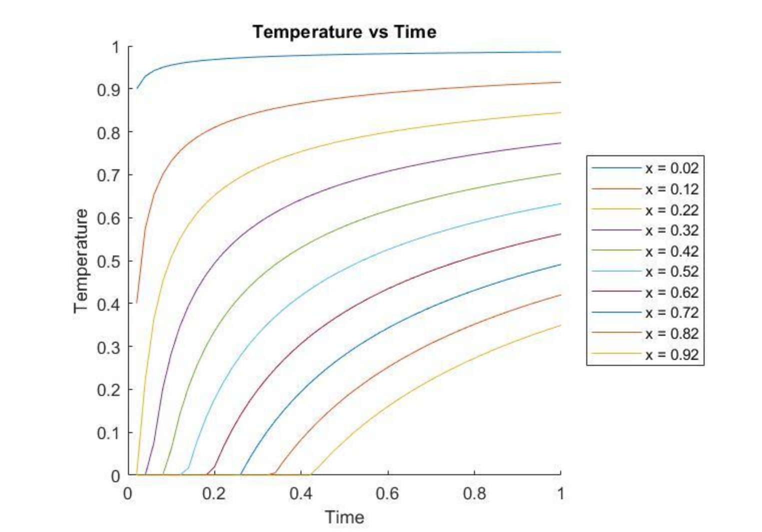

Figure 3. Solutions curves in two dimensions w.r.t. temperature and time.

Future Work

With an ODE now dependent only on ξ and subject to a stationary boundary condition in the moving frame, further analysis of the model will be able to:

- Fully incorporate the phase change boundary
- Visualize quasi-steady-state solutions
- Analyze the stability of quasi steady-state solutions (characterization of shocks in the PDE)
- Involve numerical instantiation of the solutions (finite difference schemes)