Determining the Equilibrium State of a High-Temperature Plasma in a Magnetic Confinement Nuclear Fusion Reactor

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Background and Research Motivation

Magnetic confinement fusion, as the name suggests, is a method in which hot plasmas are confined by twisted magnetic field lines to sustain a fusion reaction. One of the two most promising magnetic confinement fusion devices is the tokamak, which forces the plasma into the shape of a torus.

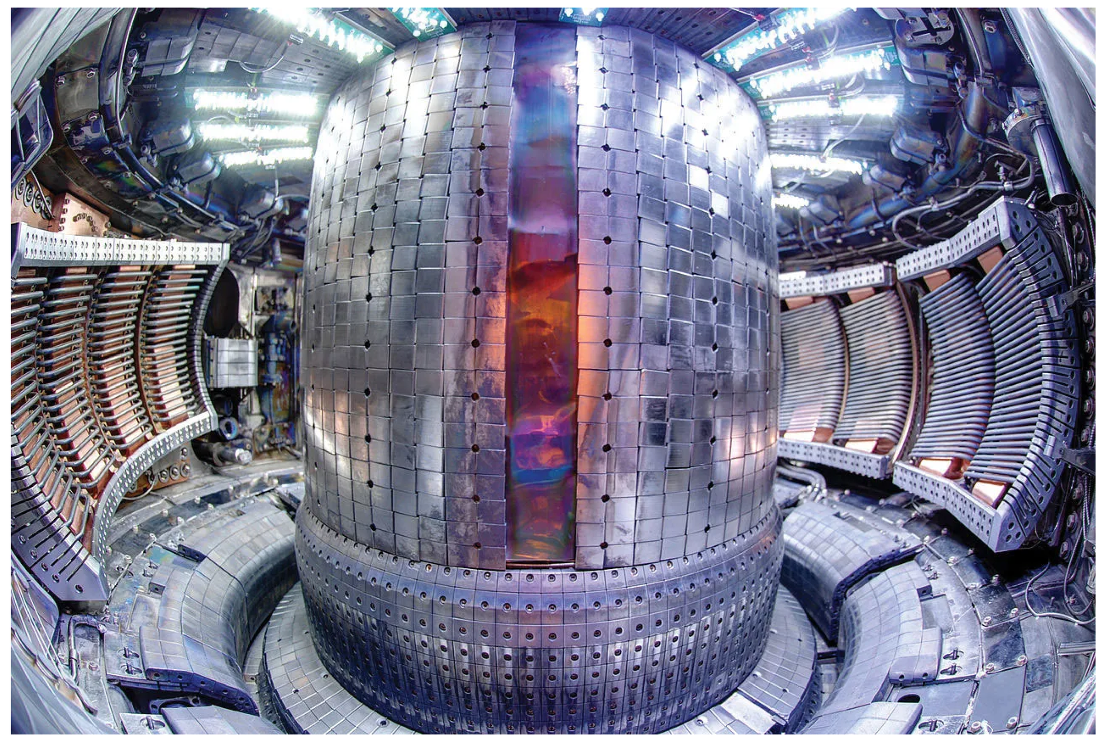

Figure 1. A photo of the interior of Alcator C-Mod, a small, decommissioned tokamak that holds the record for largest average plasma pressure at 2.05 atmospheres.

While many wave modes and instabilities that arise in tokamak plasmas are due to kinetic theory [1], the most disastrous can be avoided by studying these systems using magnetohydrodynamics (MHD). However, before any stability analyses can begin, good equilibrium solutions must be found and solved for, which serves as the motivation for this work. Here we present a successive over-relaxation solver to the Grad-Shafranov equation.

A Brief Overview of Ideal MHD

MHD can be derived by calculating the various fluid moments from the equation of state and making assumptions until the equation set is closed [2]. The set of equations for ideal MHD is,

$$\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\vec{u}) = 0 & \nabla\cdot\vec{B} = 0 \\ &\rho\left(\frac{\partial \vec{u}}{\partial t} + \vec{u}\cdot\nabla\vec{u}\right) = -\nabla p + \vec{f}_{ext} & \nabla\times\vec{E} = -\frac{\partial\vec{B}}{\partial t} \\ & & \nabla\times\vec{B} = \mu_0\vec{J} \\ &\frac{\partial e}{\partial t} + \nabla\cdot(e\vec{u}) = -p\nabla\cdot\vec{u} & \vec{E} + \vec{u}\times\vec{B} = 0, \end{aligned} \tag{1}$$

where ρ is the fluid density, $\vec{B}$ is the magnetic field, $\vec{E}$ is the electric field, $\vec{u}$ is the fluid velocity, p is the pressure, e is the internal energy, and $\vec{J}$ is the current density.

The Grad-Shafranov Equation

Using the equation set above, a partial differential equation describing equilibrium for a toroidal plasma can be derived via force balance between ∇p and the Lorentz force. The equilibrium equation in cylindrical coordinates is,

$$\Delta^*\psi = -\mu_0 R^2\frac{\mathrm{d}p}{\mathrm{d}\psi} - F\frac{\mathrm{d}F}{\mathrm{d}\psi}, \tag{2}$$

where,

$$\Delta^* = \frac{\partial^2}{\partial R^2} - \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial^2}{\partial Z^2}.$$

Δ^* is called the Grad-Shafranov operator. Physically, ψ is the poloidal magnetic flux per radian [3]. Solving this equation for ψ yields an equilibrium solution. $p(\psi)$ and $F(\psi)$ are free functions of any form. $F(\psi)$ is related to the magnetic field in the toroidal direction by $F = RB_{\mathrm{T}}$.

The Asymptotic Limit of Large Aspect Ratio

The limit of large aspect ratio serves as a good test for the successive over-relaxation code. The aspect ratio is the ratio of major radius to minor radius, or in the case of Fig. 2, R_0/a.

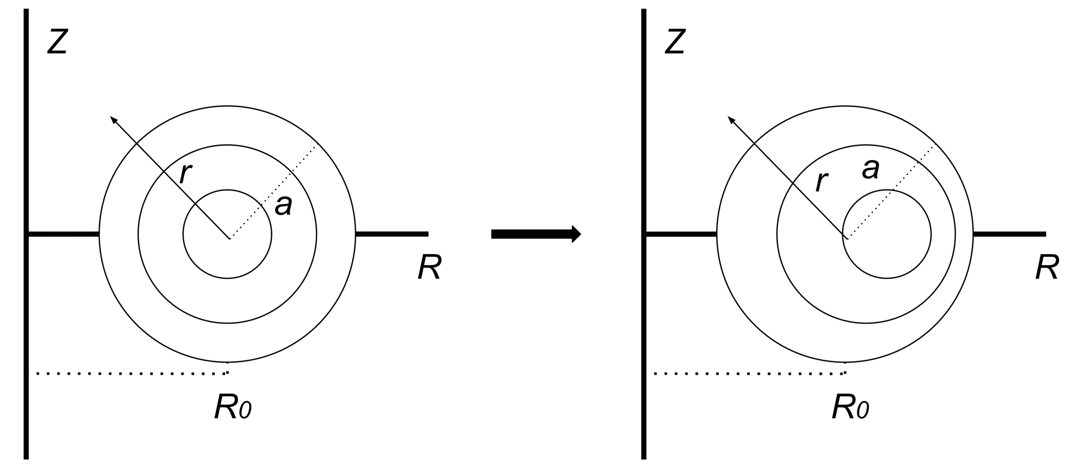

Figure 2. A diagram of how the flux surfaces of constant ψ are shifted outward as a result of the Shafranov shift. Note that $\vec{r}$, a, and R_0 are from the center of the outer most flux surface, and not necessarily from the magnetic axis.

In the limit of large aspect ratio, ψ can be expanded in powers of a/R_0 yielding,

$$\psi(r,\theta) = \psi_0(r) + \bar{\psi}_1(r)\cos\theta. \tag{3}$$

A polar coordinate system in the RZ-plane is used to define ψ [2]. The Shafranov shift is also derived by a similar expansion. The Shafranov shift can be written as,

$$\Delta(r) = -\frac{\bar{\psi}_1(r)}{R_0 B_\theta(r)}. \tag{4}$$

The exact functional form of ψ used in this test was,

$$\psi(r,\theta) = \frac{c_1}{2}r^2 + \left\{\frac{c_1 r}{R_0}\left[\frac{1}{3}(r-a) + \frac{\mu_0 R_0^2 c_2}{2c_1^2}(r^2 - a^2)\right]\right\}\cos\theta, \tag{5}$$

where $R_0 = 100$ m, $c_1 = .22$, and $c_2 = 15$. By increasing a, the radius of the outer-most flux surface, the magnitude of Δ increases.

Comparison of Computational and Analytical Solutions

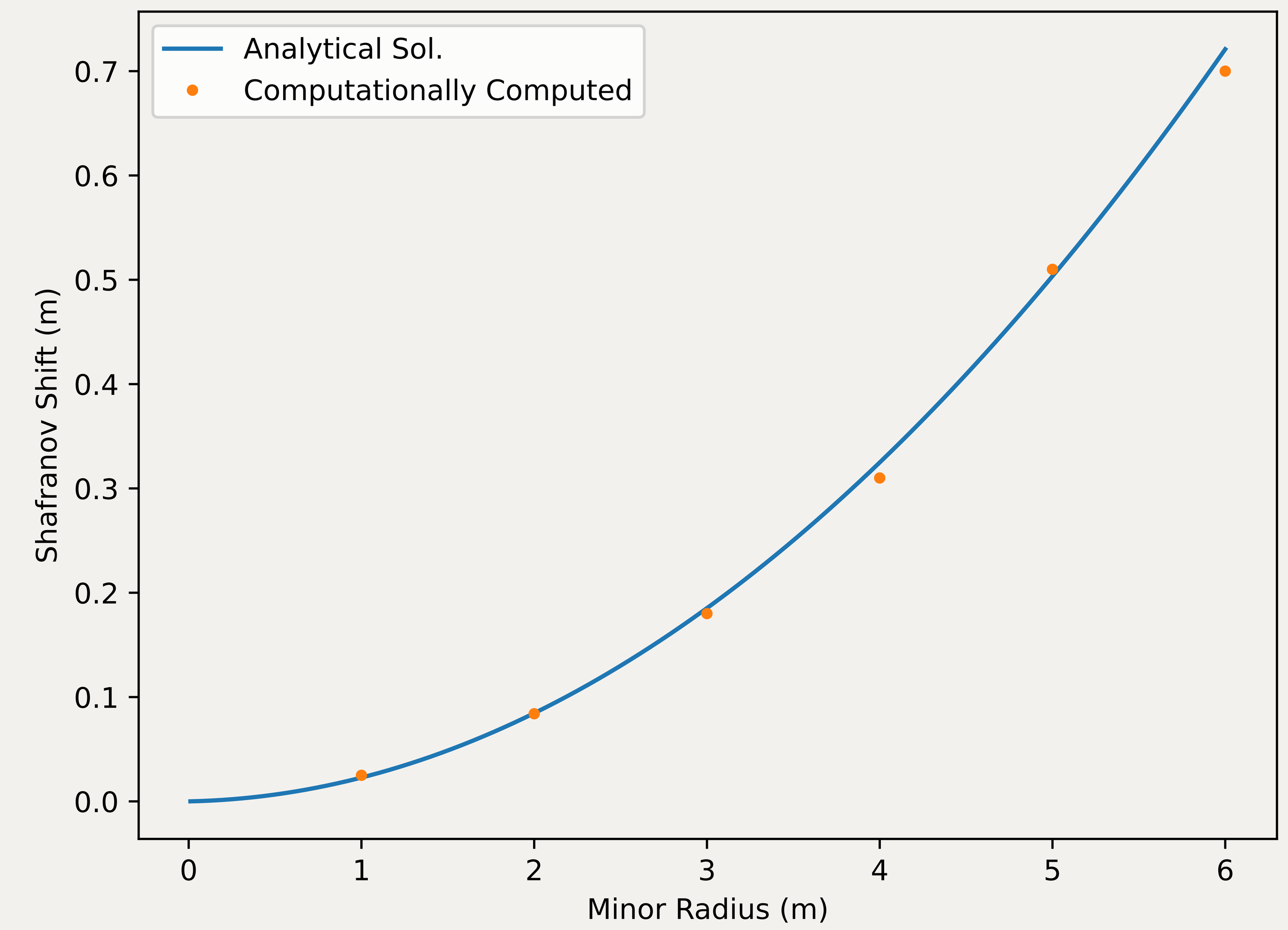

Figure 3. A plot of the Shafranov shift with respect to minor radius. The analytical solution for the Shafranov shift is shown by the blue line and the computational solutions are shown as the orange dots.

As shown in Fig. 3, the analytical solution of the Shafranov shift and the numerical solutions show great agreement for minor radii between 1 m and 6 m. These radii correspond to a range of aspect ratios that spans from 100 to 16.67. For $a > 6$, the analytical solution for $\psi(r,\theta)$ exhibits a smaller shift than Eq. 4 suggests.

An Overview of the Code

Since the Grad-Shafranov equation is an elliptic partial differential equation, a modified second-order successive over-relaxation Poisson solver was used to solve for the function $\psi(R,Z)$ [4]. Written on a discretized grid, the Grad-Shafranov equation looks like,

$$\begin{aligned} \frac{R_i}{(\delta R)^2 R_{i+1/2}}\psi_{i+1,j} - \left[\frac{R_i}{(\delta R)^2 R_{i+1/2}} + \frac{R_i}{(\delta R)^2 R_{i-1/2}} + \frac{2}{(\delta Z)^2}\right]\psi_{i,j} \\ + \frac{R_i}{(\delta R_i)^2 R_{i-1/2}}\psi_{i-1,j} + \frac{1}{(\delta Z)^2}\psi_{i,j+1} + \frac{1}{(\delta z)^2}\psi_{i,j-1} = \mu_0 R J_{i,j}, \end{aligned} \tag{6}$$

where δR and δZ are the step sizes of the grid, R_i is the distance from the symmetry axis at the point $\psi_{i,j}$, and $\mu_0 R_i J_{i,j}$ is the source function at the same grid point as $\psi_{i,j}$. The code is complete enough to test against analytical solutions to the Grad-Shafranov equation (shown in the center column). Below we discuss how modifications to this base code are made to better emulate real-world tokamaks.

Applications to Real-World Tokamaks

Tokamaks often have shaping coils and Ohmic heating solenoids to modify the shape of the plasma edge to aid in confinement [3]. To account for these shaping coils, the Green's function for an axisymmetric current can be used to find these coils' contributions to the value of ψ on the computational boundary [3, 5]. Once the successive over-relaxation code has run, critical point analysis can be done to update the plasma current density. The plasma's contribution to ψ on the computational boundary can be recalculated via the same Green's function [3, 6]. This process is then iterated until convergence is reached, as show in Fig. 4. The outer loop is called Picard iteration, the inner loop is the successive over-relaxation method described previously.

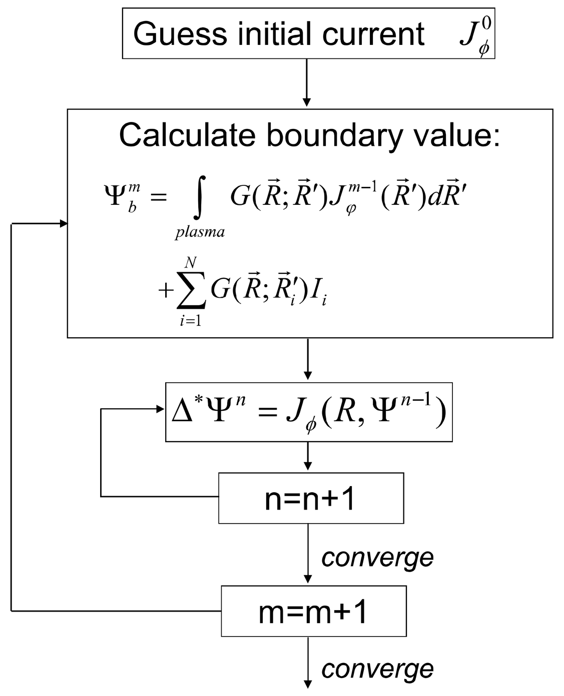

Figure 4. A flowchart showing the code's nested loop structure.

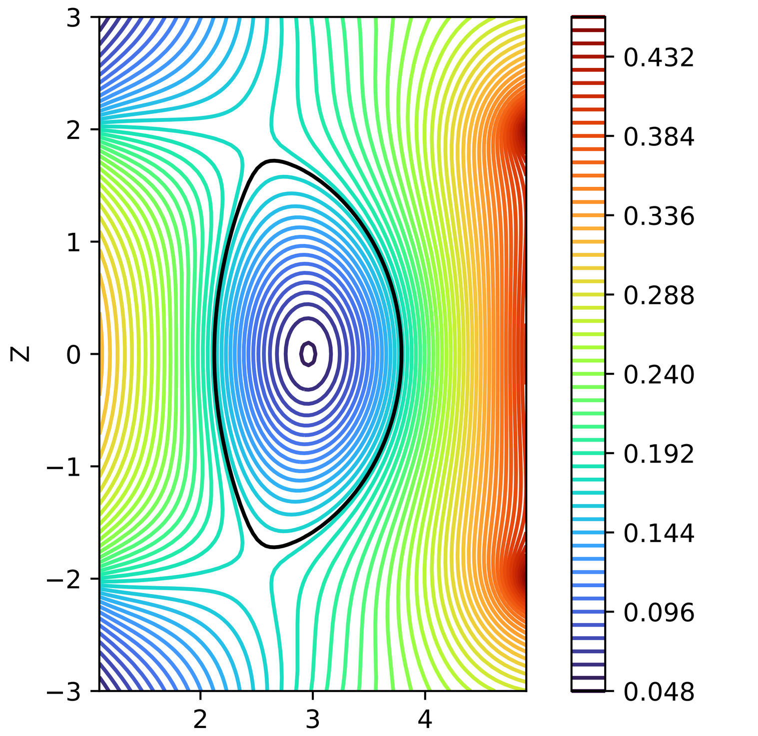

Figure 5. A free-boundary solution with an Ohmic heating solenoid and two poloidal field coils located outside of the computational boundary.

Fig. 5 shows an example D-shaped plasma cross-section in a simple tokamak with an Ohmic heating solenoid and two outer poloidal field coils. Future work with this project could be to design a coil set-up that would make for a good comparison to analytical solutions.

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