Generalized Quasilinear Simulations of 2D Strongly Stratified Kolmogorov Flow

Adhithiya Sivakumar[1], Gregory P. Chini[1]
[1]Department of Mechanical Engineering, University of New Hampshire, Durham, NH 03824, USA

Compared to DNS, how accurately does the Generalized Quasilinear (GQL) approximation reproduce the dynamics and statistics of 2D, strongly stratified Kolmogorov flow?

The GQL Approximation

Ad-hoc Quasilinear Reduction

Consider a PDE of the form

$$\boldsymbol{u}_t = L(\boldsymbol{u}) + N(\boldsymbol{u},\boldsymbol{u}); \quad \boldsymbol{u} = \boldsymbol{u}(x,z,t)$$

Decompose $\boldsymbol{u} = \overline{\boldsymbol{u}} + \boldsymbol{u}'$, and in result, keep $N(\boldsymbol{u}',\boldsymbol{u}')$ terms only where they feed back onto the mean.

$$\overline{\boldsymbol{u}}_t = L(\overline{\boldsymbol{u}}) + N(\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}}) + \overline{N(\boldsymbol{u}',\boldsymbol{u}')}$$
$$\boldsymbol{u}'_t = L(\boldsymbol{u}') + N(\overline{\boldsymbol{u}}, \boldsymbol{u}') + N(\boldsymbol{u}',\overline{\boldsymbol{u}}).$$

The mean field equation is nonlinear, but the $\boldsymbol{u}'$ equation is linear w.r.t the mean.

Generalization

Linearize w.r.t. **slowly varying** mean. In practice, this amounts to a redefinition of $\overline{\boldsymbol{u}}$ and $\boldsymbol{u}'$ using a spectral filter.

$$\overline{\boldsymbol{u}} \equiv \sum_{|k| \leq \Lambda} \boldsymbol{u}_k e^{ikx}, \qquad \boldsymbol{u}' \equiv \sum_{|k| > \Lambda} \boldsymbol{u}_k e^{ikx}$$

In the decomposed system, discard triadic interactions in energy-conserving manner.

$$\overline{\boldsymbol{u}}_t = L(\overline{\boldsymbol{u}}) + \overline{N(\overline{\boldsymbol{u}}, \overline{\boldsymbol{u}})} + \overline{N(\boldsymbol{u}',\boldsymbol{u}')}$$
$$\boldsymbol{u}'_t = L(\boldsymbol{u}') + N'(\overline{\boldsymbol{u}}, \boldsymbol{u}') + N'(\boldsymbol{u}',\overline{\boldsymbol{u}}).$$

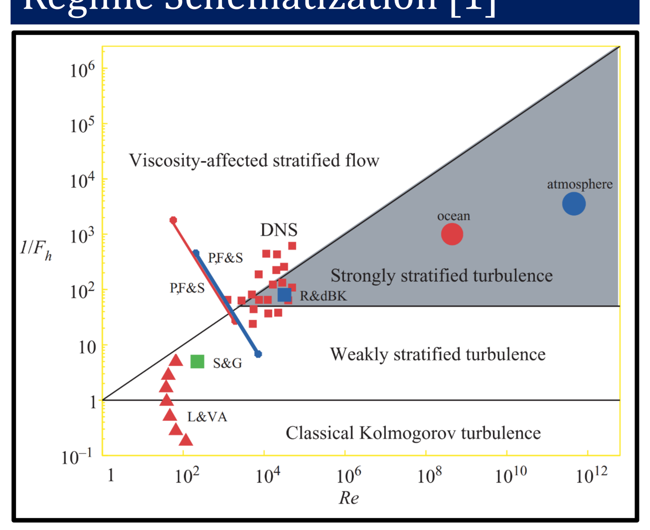
GQL, like QL, works by restricting nonlinear interactions between modes. However, unlike QL, GQL allows non-local energy exchange between high modes ($\boldsymbol{u}'$) by scattering off low modes ($\overline{\boldsymbol{u}}$). Note that the cut-off wavenumber, Λ, effects a homotopy between QL ($\Lambda = 0$) and fully nonlinear ($\Lambda \to \infty$) dynamics.

Strongly Stratified Turbulence

At A Glance

$$Fr = \frac{U}{NL} \ll 1, \qquad Re = \frac{UL}{\nu} \gg 1, \qquad Re_b = ReFr^2 \gg 1.$$

Regime Schematization [1]



Phenomenology [1]

Strongly stratified and turbulent flows exhibit a high degree of spatial scale-separation, which can be exploited to asymptotically derive the QL/GQL reduced equations in the large Re_b and small Fr limit. This provides an alternative to DNS of these flows, which is often computationally expensive on geo-physical scales.

Equations, Scales, Parameters

2D Anisotropically Scaled Boussinesq Equations

$$u_t + uu_x + wu_z = -p_x + \frac{1}{Re_b}\left(Fr^2 u_{xx} + u_{zz}\right) + \frac{m^2}{Re_b}\cos(mz)$$

$$Fr^2(w_t + uw_x + ww_z) = -p_z + b + \frac{Fr^2}{Re_b}\left(Fr^2 w_{xx} + w_{zz}\right)$$

$$b_t + ub_x + wb_z = -w + \frac{1}{\mathrm{Pr}\, Re_b}\left(Fr^2 b_{xx} + b_{zz}\right)$$

$$u_x + w_z = 0$$

Characteristic Scales

$$x \sim L, \qquad z \sim \frac{U}{N}, \qquad u \sim U, \qquad t \sim \frac{L}{U},$$

$$p \sim \rho_0 U^2, \qquad b \sim UN, \qquad w \sim UFr.$$

Parameters

- $\Lambda = 0, 1, 4, 16, 32$
- $N_x = N_z = 512$
- $L_x = 1\ (50\ U/N)$
- $L_z = 4\pi/3$
- $Fr = 0.02$
- $Re_b = 50$
- $\mathrm{Pr} = 1$
- $m = 3$
- $t_{end} = 50$

Linear Dynamics

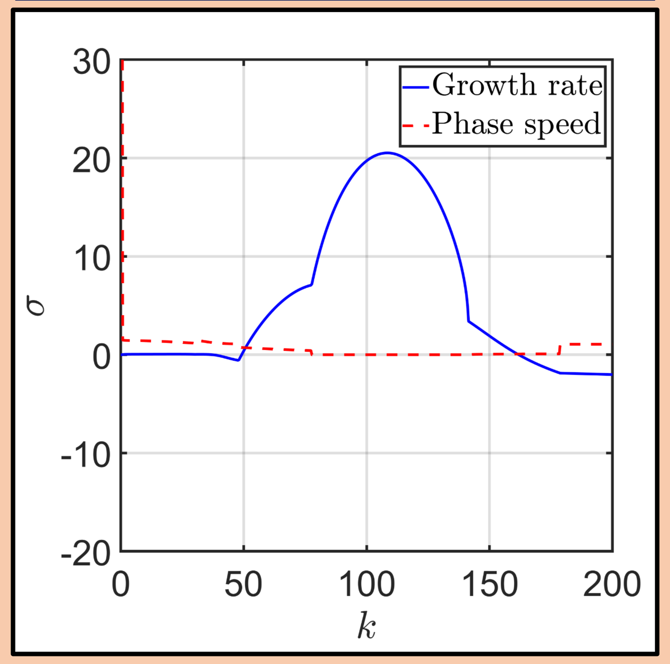
Basic State

$$U_b = \cos(mz), \quad W_b = 0, \quad B_b = 0$$

Instability Criteria

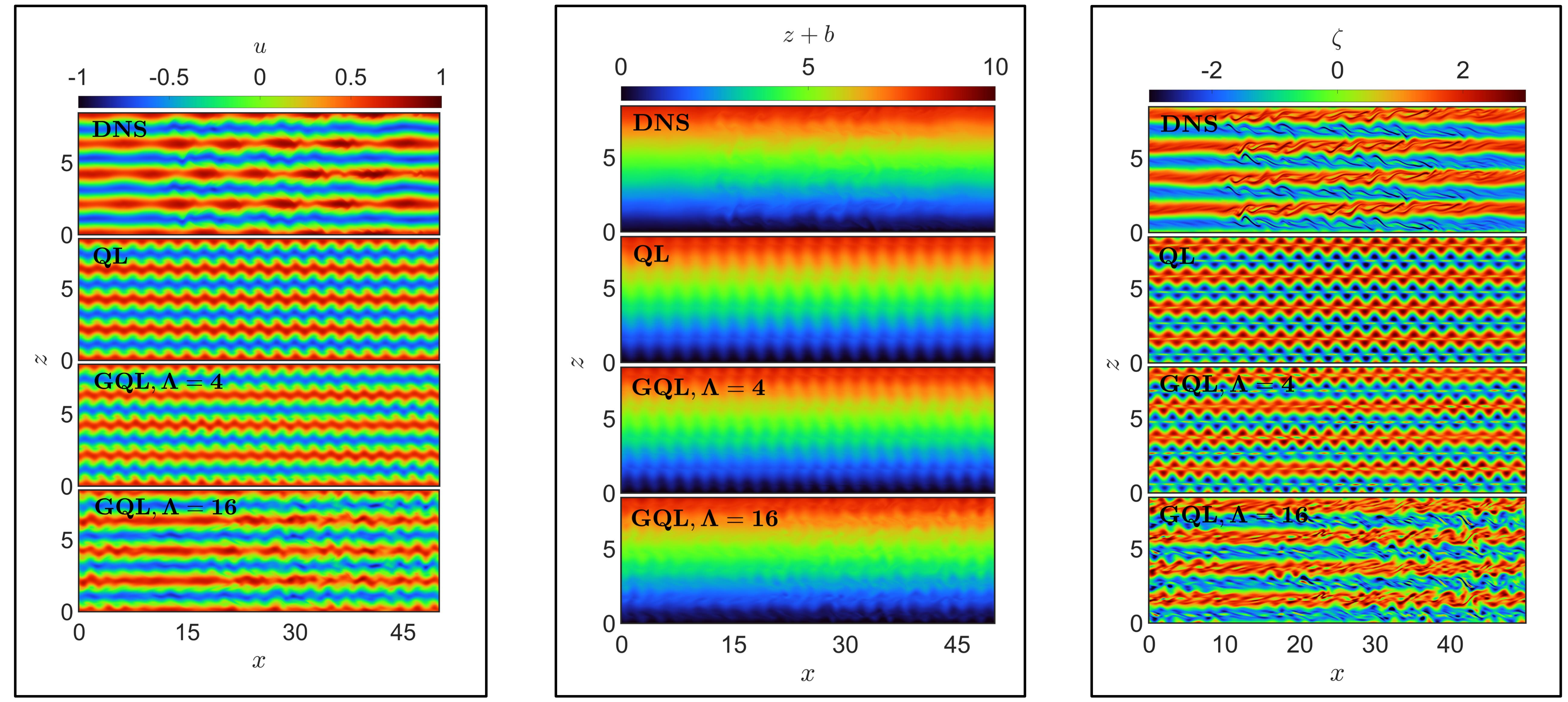
$$Ri_g \equiv \frac{1}{\mathrm{m}^2 \sin^2(mz)} < \frac{1}{4}$$

Growth Rate Curve

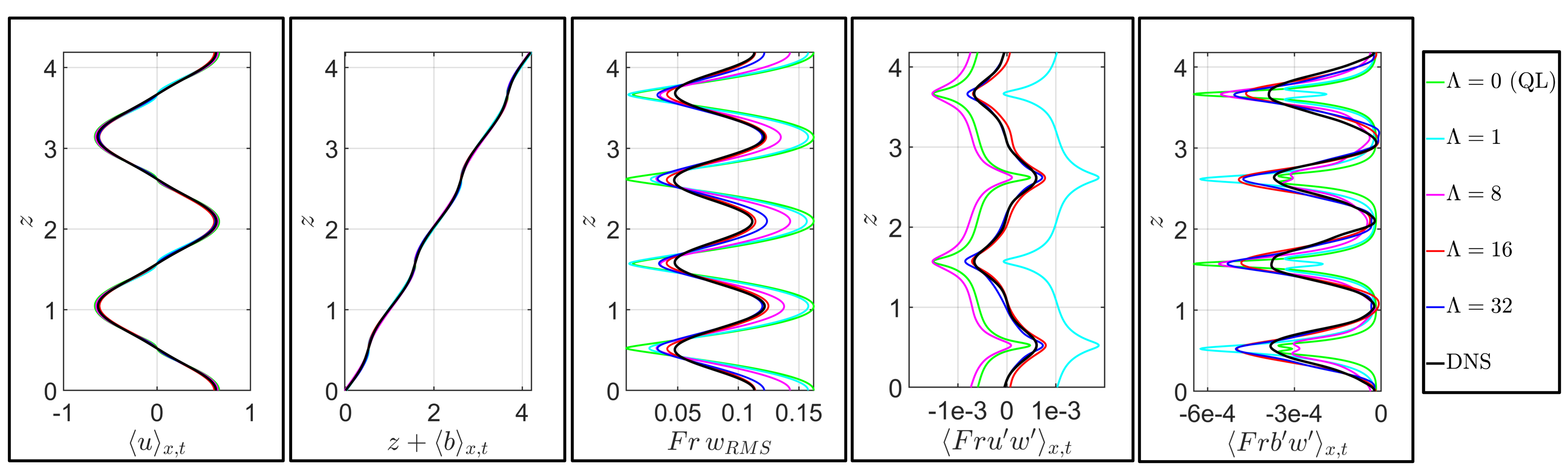


Linear stability analysis reveals two distinct growth rate peaks: a fast, small-scale instability with zero phase speed (Kelvin-Helmholtz), and a slow, large-scale instability (not visible in plot) with non-zero phase speed. Interaction between these two linearly unstable modes informs the fully nonlinear dynamics observed via DNS.

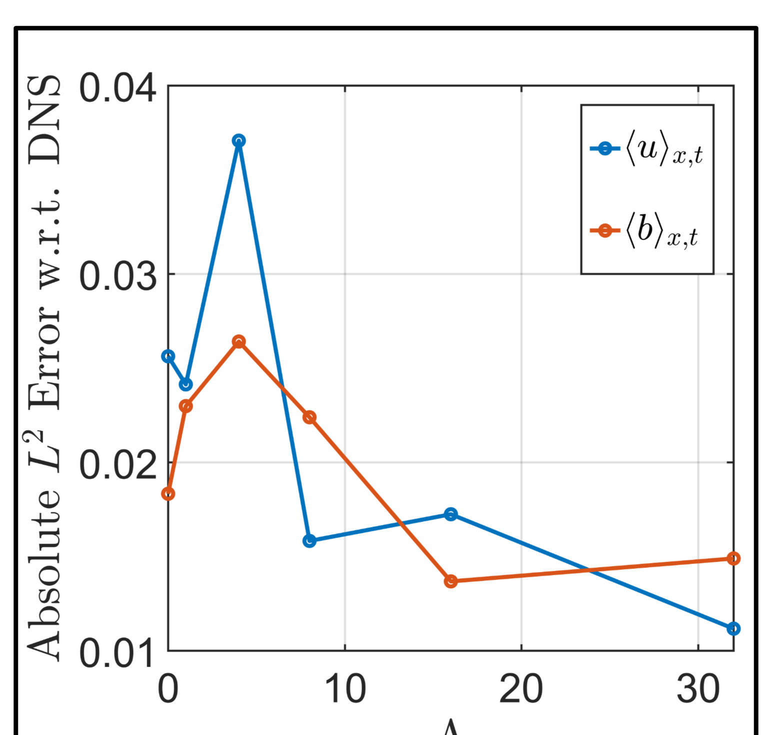
Nonlinear Dynamics: Snapshots at $t \approx 50$



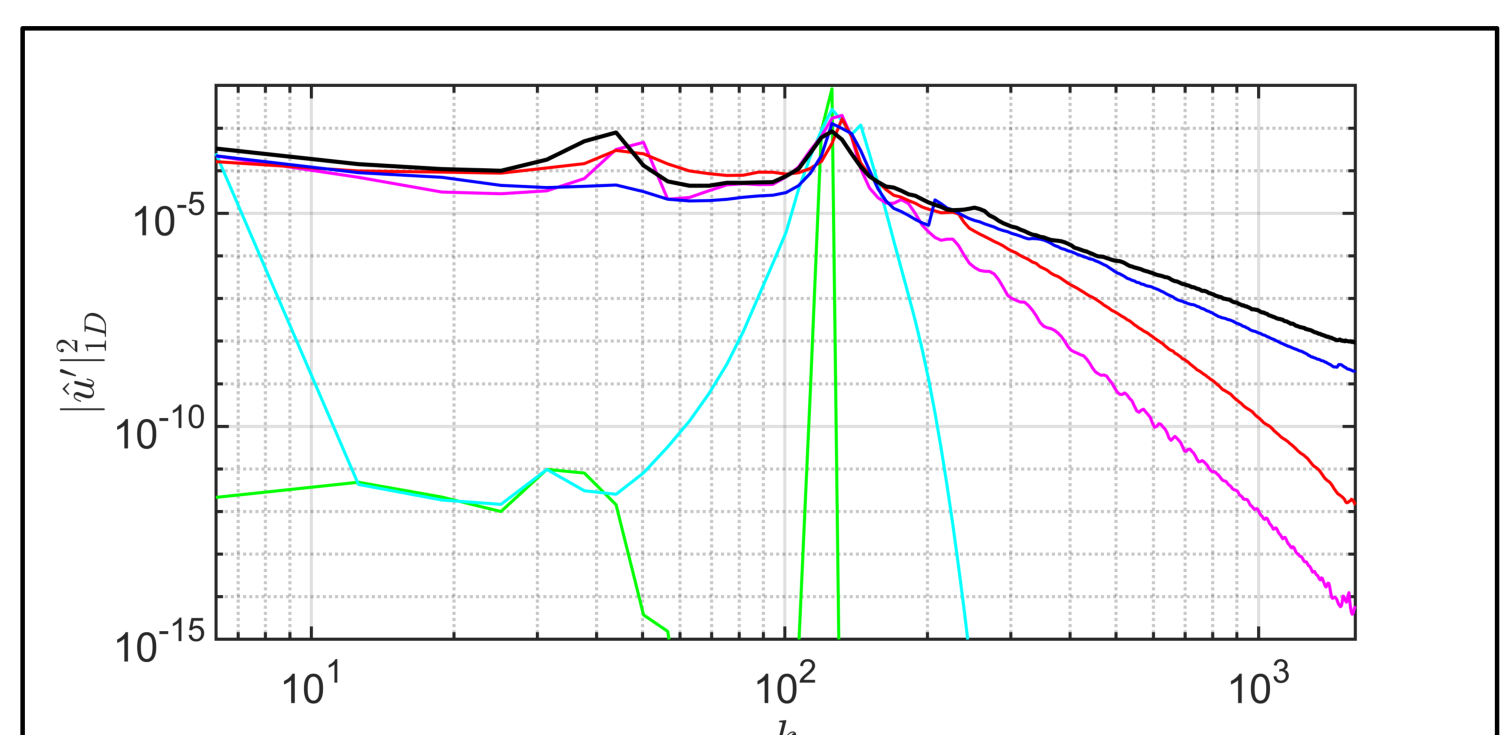
Comparison of Turbulent Statistics (Averaged over 1500 Buoyancy Periods)



L^2 Error Estimates



1D Horizontal Kinetic Energy Spectra



Observations

- Local maxima in the DNS horizontal KE spectrum, corresponding to local maxima in the linear growth rate curve, indicate that the nonlinear dynamics are dominated by the interaction of features with two distinct horizontal length scales.
- All computed GQL spectra reproduce the large-wavenumber peak, and therefore the dominant small-scale features of the flow.
- However, the small-wavenumber peak is suppressed in GQL spectra with small cut-off wavenumber ($\Lambda < 8$), corresponding to the diminished large-scale modulation of small-scale structures in these runs.
- This accounts for the visual and statistical discrepancies between GQL and fully nonlinear dynamics, as well as the non-monotonic increase in accuracy (relative to DNS, as measured in the L^2 norm) of GQL mean quantities with Λ.
- Notably, GQL with $\Lambda = 16$ accurately reproduces most turbulent statistics and exhibits visual similarity to fully nonlinear dynamics.

Future Work

- Analysis of DNS and GQL nonlinear transfer spectra in order to identify important energy pathways, explain GQL-DNS discrepancies more precisely, and propose an optimal cut-off wavenumber Λ.
- GQL simulations which contain a small number of dynamical active modes (possibly corresponding to the spectral peaks), with the rest of the modes slaved to them.
- Extension to 3D and tall domains.

References

[1] Brethouwer, G., Billant, P., Lindborg, E. & Chomaz, J.-M. 2007. Scaling analysis and simulation of strongly stratified turbulent flows. *J. Fluid Mech.* **585**, 343-368.