Tropical Calabi-Yau Mirrors in non-Fano Toric Varieties

Michael J.S. Lathwood†

Advisor: Per Berglund

†University of New Hampshire

String theory $\implies$ mirror symmetry

A mathematical phenomenon known as mirror symmetry was first discovered by physicists in the late 80's by studying supersymmetric quantum field theories [5] arising from *string compactifications*.

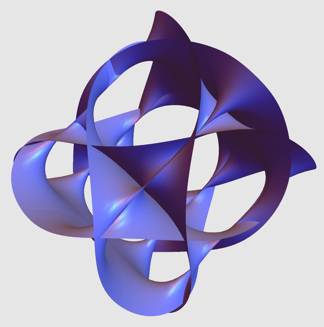

Figure 1. A 2-dimensional cross section of a 6-dimensional Calabi-Yau manifold X.

To study string compactifications, physicists construct 6-dimensional spaces called *Calabi-Yau manifolds*. The geometry and topology of the manifold determine the physics of the resulting theory. One would like to construct a space that produces phenomenologically consistent results, but this can be very complicated as the Calabi-Yau is sensitive to many parameters (called moduli). Mirror symmetry allows us to construct so-called *mirror pairs* $(X, \check{X})$ of Calabi-Yaus. Mirror spaces are geometrically and topologically distinct, but there is a duality between their moduli, so the physics of the resulting string compactifications is the same. One of the mirrors is often much easier to work with than the other, so understanding mirror symmetry is very important for constructing physically relevant string theories.

Conjecture 1: Homological mirror symmetry

Kontsevich's homological mirror symmetry (HMS) conjecture [8] *has been proven for various cases* [2]*. HMS states that for a there should be an equivalence of algebraic structures associated to a mirror pair $(X, \check{X})$. More technically, the HMS can be stated as a quasi-equivalance of derived A_∞ categories:*

$$\mathfrak{D}_A(X) \simeq \mathfrak{D}_B(\check{X}) \tag{1}$$

- *$\mathfrak{D}_A(X)$: boundary conditions that preserve A-type supersymmetry on X*
- *$\mathfrak{D}_B(\check{X})$: boundary conditions that preserve B-type supersymmetry on $\check{X}$*

A boundary condition in string theory is described by a brane. Roughly, an A-brane is a Lagrangian submanifold $L \subset X$ and a B-brane is a coherent sheaf $\mathcal{O} \twoheadrightarrow \check{X}$.

How to construct toric varieties $\mathcal{Y}_\Delta$ from polytopes Δ

A toric variety is a space that is described by the zero set of some polynomial and contains an algebraic torus as a dense subset

1. Let $M \cong \mathbb{Z}^{n+1}$ be an $n+1$ dimensional lattice of points and let $N = \mathrm{Hom}(M, \mathbb{Z}) \cong \mathbb{Z}^{n+1}$ be the dual lattice.
2. Specify a (spanning) polytope Δ^* by the hull of some vertices $\nu_\rho \in N$
$$\Delta^* \subset N \otimes \mathbb{R} \cong \mathbb{R}^{n+1} \tag{2}$$
3. Construct the dual (Newton) polytope $(\Delta^*)^\circ = \{\mu \in M_\mathbb{R} | \mu \cdot \nu \geq -1 \;\; \forall \nu \in \Delta^*\} \subset M \otimes \mathbb{R}$
4. The normal fan $\Sigma^{(1)}$ consists of the rays that are orthogonal to the faces of $(\Delta^*)^\circ$
5. Use $\Sigma^{(1)}$ to describe the torus T^{n+1} orbits in an $(n+1)$-dimensional toric variety $\mathcal{Y}_\Delta$

In other words, Δ^* completely determines the geometry and topology of $\mathcal{Y}_\Delta$.

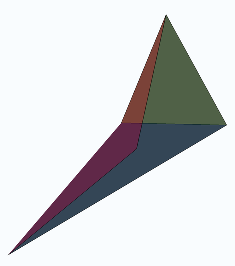

Figure 2. The spanning polytope Δ^* for the generalized Hirzebruch 3-fold $\mathcal{F}_3^{(3)}$

How to construct Calabi-Yau hypersurfaces $X \hookrightarrow \mathcal{Y}_\Delta$

A Calabi-Yau hypersurface is a codimension 1 manifold with vanishing Ricci curvature

1. Each vertex $\nu_\rho \in \Delta^*$ corresponds to a codimension 1 divisor D_ρ.
2. The anticanonical divisor $D_0 = -K_{\mathcal{Y}_\Delta} := K^*$ of the toric variety $\mathcal{Y}_\Delta$ is given by the sum of the toric divisors
$$D_0 = \sum_{\rho \in \Sigma^{(1)}} D_\rho \tag{3}$$
3. By the adjunction formula, $K_{D_0} = (K_{\mathcal{Y}_\Delta} + D_0)\big|_{D_0} = K_{\mathcal{Y}_\Delta} - K_{\mathcal{Y}_\Delta} = 0$
4. Therefore *an anticanonical section is a Calabi-Yau n-fold X*, since having trivial canonical class is equivalent to having vanishing Ricci curvature.

The geometric phase of a GLSM with D-flat configurations given by $\mathcal{Y}_\Delta$ is a nonlinear sigma model on X, i.e. a string compactification.

The Fano condition

The toric variety $\mathcal{Y}_\Delta$ is Fano if all the D_ρ in K^* have positive coefficients

Hori and Vafa originally extended mirror symmetry of SCFTs given by a nonlinear sigma models on Calabi-Yaus to SCFTs given by Landau-Ginzburg (LG) models [6]. An LG model is given by a noncompact space $M \otimes \mathbb{C}^* := M_{\mathbb{C}^*}$ with a complex valued function called the superpotential W_z.

$$W_z(Y) = \sum_{\rho \in \Sigma^{(1)}} z_{i(\rho)}^{\lambda(\rho)} Y^\rho \tag{4}$$

A mirror to a Calabi-Yau $X \hookrightarrow \mathcal{Y}_\Delta$ can be constructed from this LG data as

$$\check{X}_z = W_z^{-1}(1). \tag{5}$$

1. z_i are complex structure moduli of $\check{X}_z$ which control the "shape" of the space
2. t_i are the Kähler moduli of X which control the "size" of the space
3. The limit where all $t_i \to \infty$ and all $z_i \to 0$ is called the large radius/large complex structure limit (LR/LCSL), and in this limit $t_i = -\log z_i$

Strictly speaking, Hori-Vafa mirror symmetry should only work when $\mathcal{Y}_\Delta$ is Fano, and my research aims to extend this using HMS.

Theorem 2: HMS for $X \hookrightarrow \mathcal{Y}_\Delta$

The following diagram of categorical embeddings, projections and equivalences commutes [1]*.*

$$\begin{tikzcd} D^b\mathsf{Coh}(\mathcal{Y}_\Delta) \arrow[r, leftrightarrow, "\mathsf{HMS}"] \arrow[d, shift left, hook] & D^b\mathcal{FS}(M_{\mathbb{C}^*}, W_z) \arrow[r, leftrightarrow] & D^b\mathcal{FS}(\bar{M}_{\mathbb{C}^*}, \bar{W}_z) \arrow[d, shift left] \\ D^b\mathsf{Coh}(X) \arrow[u, shift left] \arrow[rr, leftrightarrow, "\mathsf{HMS}"] & & D^b\mathcal{F}(\check{X}_z) \arrow[u, shift left, hook] \end{tikzcd} \tag{6}$$

We will show the bottom arrow holds in an example where $\mathcal{Y}_\Delta$ is non-Fano.

Tropical geometry

Consider the map

$$\begin{aligned} \mathsf{Log}_z : M_{\mathbb{C}^*} &\longrightarrow M_\mathbb{R} \\ (Y_1, \ldots, Y_{n+1}) &\mapsto (\log_z |Y_1|, \ldots, \log_z |Y_{n+1}|). \end{aligned} \tag{7}$$

The topical amoeba $\mathcal{A}(C)$ of $C \subset M_{\mathbb{C}^*}$ is given by [9]

$$\mathcal{A}(C) = \lim_{z \to 0} \mathsf{Log}_z\, C. \tag{8}$$

Theorem 3: Tropical period integral formula

I proved the following integral formula for Calabi-Yau 3-folds $X \hookrightarrow \mathcal{Y}_\Delta$ in the LR/LCSL.

$$\begin{aligned} \int_X \widehat{\Gamma}_X e^{\sum_a t_a \omega_a} = \int_{\check{C}_z^+} \check{\Omega}_z = &-\zeta(3)\left(\frac{1}{3!}K^* \sum_{\nu \neq \rho_1 \neq \rho_2} D_\nu D_{\rho_1} D_{\rho_2} - \frac{1}{2!}(K^*)^2 \sum_{\nu \neq \rho} D_\nu D_\rho\right) \\ &- \frac{1}{2!}\zeta(2)(K^*)^2 \sum_{\nu \neq \rho}(-\log z_{a(\rho)}) D_\nu D_\rho + \frac{1}{3!}K^*\left(\sum_{a=1}^{s}(-\log z_a) D_a\right)^3 \end{aligned} \tag{9}$$

Here, $\check{C}_z^+ \subset \check{X}_z$ is an A-brane with the property $\mathcal{A}(\check{C}_z^+) \cong (\Delta^*)^\circ$. These results, which use tropical geometry, match with the "classical" results [7]*. Now, $\mathcal{Y}_\Delta$ need not be Fano for this to hold.*

A non-Fano example

The m-twisted Hirzebruch $(n+1)$-fold $\mathcal{F}_m^{(n+1)}$ is a fibration of $\mathbb{C}P^n$ over $\mathbb{C}P^1$. For $n = 1$, the anticanonical class is given by a sum of two toric divisors, where each one corresponds to a $\mathbb{C}P^1$.

$$K^* = 2D_1 + (2 - m)D_2 \tag{10}$$

For $m = 3$, we can see that $\mathcal{F}_3^{(2)}$ is non-Fano. This can also be seen by looking at the normal fan

$$\Sigma^{(1)} = \{(1,0), (0,1), (-1,0), (-m,-1)\}, \tag{11}$$

which is non convex for $m > 2$ due to the $(-1, 0)$ point. However, I showed an equation analogous to Equation (9) still holds, so $\check{X}_z$ is still mirror to X.

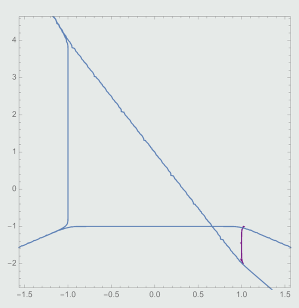

Figure 3. The tropical amoeba $\mathcal{A}(\check{C}_z^+)$ for $\mathcal{F}_3^{(2)}$.

Extending mirror symmetry

With my period integral formula, we hope to extend our result to a statement about categories. We are doing this by constructing $\mathcal{A}(\check{C}_z^+) \cong (\Delta^*)^\circ$ and matching with well-studied cases [3, 4].

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