Prospective Secondary Mathematics Teachers' Understanding of the Role of Examples in Proving

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RESEARCH PROBLEM AND MOTIVATION

- Secondary teachers' knowledge of reasoning and proving is fundamental in supporting students' meaningful participation in proof-related practices.
- Research identified gaps in teacher understanding of deductive reasoning, such as misunderstanding the roles of examples and counterexamples in proving and disproving (Tsamir et al., 2008; Ko, 2010).
- Intervention studies addressing these knowledge gaps are scarce (Stylianides, Stylianides & Weber, 2017).

RESEARCH QUESTIONS

- What specific difficulties related to understanding the roles of examples in proving do PSTs encounter in the context of geometry (quadrilaterals)?
- How can PSTs be supported in developing appropriate conceptions about the role of examples in proving?

SETTING OF THE STUDY

- ***Capstone Course Mathematical Reasoning and Proving for Secondary Teachers*** (Buchbinder & McCrone, 2020)

Objectives: Enhance future teachers' content and pedagogical knowledge for integrating reasoning and proving in their classroom instruction, for any content or grade level

Crystalize
Increase PSTs' awareness of the importance of the **logical aspects** of proof; Refresh and enhance PSTs' own content knowledge specific to proof

Connect
Increase PSTs' awareness of **student difficulties** with proving; Learn to identify ***within regular school curricula*** opportunities for proof integration

Apply
Equip PSTs with **pedagogical tools** to create or modify mathematical tasks that **integrate reasoning and proof**

Building Mathematical and Pedagogical Knowledge for Teaching Proof

The course included four modules, each dealing with a specific proof theme

- **Quantification and the Role of Examples**
- Conditional Statements and Logical Equivalence
- Direct Proof, Analyzing and Critiquing Arguments
- Indirect Reasoning and Proof by Contradiction

Participants

N=45 PSTs, seniors, with pre-requisite coursework in both proof and geometry.

Data sources

- The data came from one experience: *What can you infer from This Example?* (Buchbinder, Ron, Zodik, & Cook, 2016)
- The PSTs individually completed the experience online, in Qualtrics, prior to a whole class discussion.

METHODS/RESULTS

What can you infer from This Example?

Part 1: Given a statement, determine whether it is true or false, and justify your reasoning.

- The PSTs responded to this question before and after examining hypothetical student work.

A quadrilateral whose diagonals are congruent and perpendicular to each other is a kite.

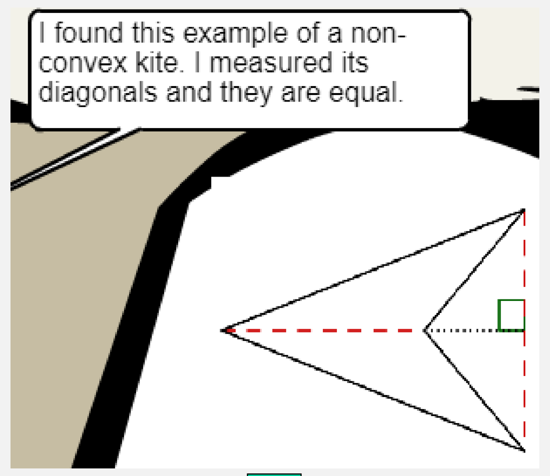
True   False ✓

Results Part 1

	Justification	Before Examining student work	After examining student work
True	"A square is a type of kite"; "Non-convex kite satisfies the statement"	5	1
False	**Correct or Unspecified Counterexample** "There are other quadrilaterals that have these properties"	7 →	22
	Irrelevant Counterexample "Square/Rhombus"	8	2
	Incorrect explanation "kites don't have congruent diagonals"	24 →	7
	Both Correct & Irrelevant Counterexamples	1	12

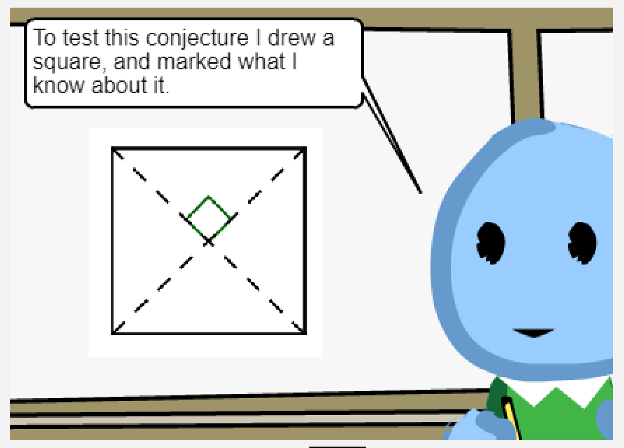
- The justifications shifted after examining student work. The number of **correct** or unspecified counterexample **increased** from 7 to 22.
- The number of **incorrect** explanation **decreased** from 24 to 7 after seeing student work.
- The number of referencing **both** correct counterexample & irrelevant example increased from 1 to 12 after going through the survey.

Example: *"Although sometimes a kite can have diagonals that are congruent and perpendicular to each other, a kite does not always have that and there also are many other quadrilaterals that have these properties. These are all counterexamples that disprove the statement."*



Example of a PST's **correct** response:
Answer: Only supports the statement

Justification: The student gives a specific example in which the conjecture holds true for a non-convex kite with congruent diagonals.



Example of a PST's **incorrect** response
Answer: Disproves the statement

Justification: Green has given a counter example to the statement. They say that a square also shares these properties and in doing so could change the original statement to ... A quadrilateral with congruent and perpendicular diagonals could be a kite or a square.

METHODS/RESULTS

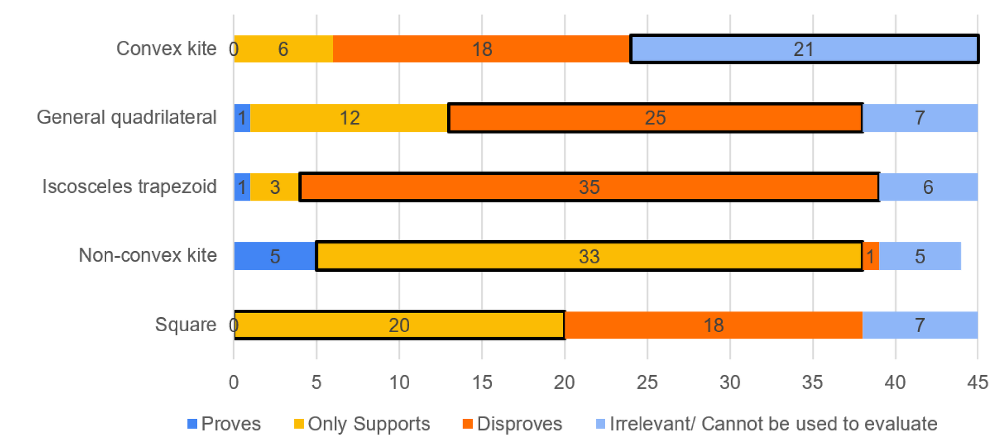
Part 2: Given five examples by hypothetical students.

Student's quadrilateral	Type of example / Expected response
A square	Supportive
Non-convex kite	Supportive
Convex Kite	Irrelevant
An isosceles trapezoid	Counterexample
General quadrilateral	Counterexample

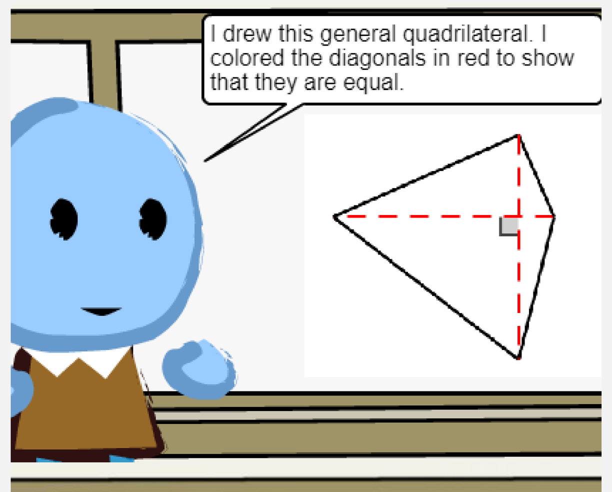
For each student example, determine its role in proving / disproving the statement and justify the answer.

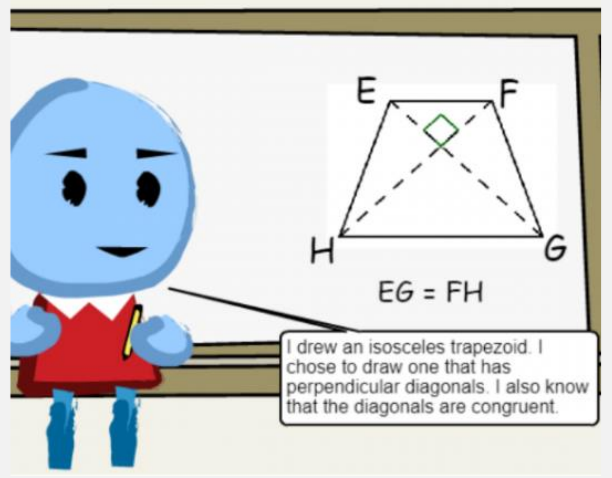
- Proves the statement is true
- Only supports the statement
- Disproves the statement
- Cannot be used to evaluate the statement

Results Part 2



- For each quadrilateral, the modal response (in a black frame) is the correct one.
- 21 PSTs (46%) correctly identified the convex kite as irrelevant, 18 PSTs (40%) incorrectly thought it disproves the statement.
- 35 PSTs (78%) correctly identified that an isosceles trapezoid is a counterexample as well as 25 PSTs (56%) correctly identified the general quadrilateral was a counterexample.
- 20 PSTs (44%) correctly identified a square is a counterexample, but 18 PSTs (40%) said it disproves the statement.





SUMMARY

Analysis of the PSTs Feedback to the Students (N=221)

1. **Perspective**: Whether a PST addressed their feedback to the student directly (you), collectively (we) or in the third person (he/she/they).
2. **Questions:** Whether the feedback included questions in their feedback
3. **Explanations:** Whether the feedback included mathematical explanation
4. **Action:** Whether the feedback is "actionable", helps to further student understanding or "not actionable".

Results of Feedback Codes

- 93 responses (42%) used the first person, and 94 (43%) used the third person.

  Example: *I would ask green why he decided to test a square?*
- 179 PST responses (81%) contained a question
  - Of these, 110 responses (61%) contained a guiding question to help the student understand what to do next to disprove the statement.

    Example: *I would again ask* ***them why this is relevant to the original statement*** *and ask them to relate it back and then draw a conclusion about the statement.*
- Only 23 responses (10%) contained some sort of mathematical explanation.

  Example: *I would try to explain to the student that when trying to prove a statement to be true, you must use arbitrary or general examples. however, when trying to disprove a statement, it is okay to only find one specific example that does not fit the guidelines of the statement*
- 135 of feedback (61%) were categorized as "actionable"

  Example of non-actionable feedback: *I would say that they were creative in their thinking. Especially going with a concave quadrilateral.*

CONCLUSIONS

- The experience provided the PSTs an opportunity to reflect on mathematical, logical and pedagogical aspects related to engaging students with proving.
  - Examining hypothetical student work helped PSTs to adjust their **mathematical and logical** responses towards the correct ones.
  - The PSTs approximated the **pedagogical** practices of interpreting student work and providing instructional feedback.
- PST interpretation of student work revealed two main difficulties in their understanding of the role of examples in proving:
  - Misunderstanding of the **logical** structure of the statement.
  - Misunderstanding of **geometry -** properties and relationships between families of quadrilaterals.

Next Steps

- Characterize PSTs' responses by locating the source of difficulty.
- Understanding fluctuations in individual PST's responses as they progress through *What can you infer from This Example?* experience.

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